GYROSCOPIC TRACKING SYSTEMS IN THE PRESENCE OF MULTIDIMENSIONAL RANDOM DISTURBANCES

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Gyroscopic tracking systems may be designed to follow multidimensional input signals. Thus, for example, on the basis of a three-axis gyroscope system [1] it is possible to construct a tracking system to follow a three-dimensional signal on three axes properly oriented in a three-dimensional space. It is possible to follow two-dimensional input signals with a two-axis gyroscope system.

On the basis of the theory of multidimensional random processes [2-5], we treat below the problem of constructing an optimal gyroscopic tracking system for following a two-dimensional input signal in the presence of not only noise in the input, but also disturbances caused by the motion of the object on which the tracking system is mounted. As is shown in the paper, these disturbances cause intercorrelation of the reduced input signals which determine the optimal weighting function, even in the case when the input signals themselves are uncorrelated. The correlation function of the reduced input signals is determined both by the statistical characteristics of the disturbances and by the structure and parameters of the transfer function matrix of the gyroscope system. Hence, the optimal weight function of the entire tracking system as a unit, and not only its correction circuit, depends to a large extent on the dynamic characteristics of the gyroscope system.

1. Gyroscopic tracking system with feedback. Error in reproducing the true input signal. A tracking system designed for following two-dimensional input signals consists of a gyroscope with three degrees of freedom and an input signal converter, the transfer function of which should be chosen so that the mean square error in reproducing the true input signal will be a minimum.

The equations of motion of a gyroscope have the following form

$$\begin{aligned} A\alpha'' + \sigma\alpha' - H\beta' &= -l[y_2(t) + \psi_2(t)] \\ B\beta'' + H\alpha' &= S[y_1(t) + \psi_1(t)] \end{aligned} \tag{1.1}$$

Here α is the angle of rotation of the external Cardan ring of the gyroscope, β is the angle of rotation of the housing of the gyroscope, H is the kinetic moment of the gyroscope, A is the moment of inertia of the gyroscope together with the housing and outer gimbal about the axis of this gibmal, B is the moment of inertia of the gyroscope together with the housing about the axis of the housing, $-\sigma\alpha'$ is the moment of the friction force in the bearings of the axle of the external gimbal, $-l\psi_2(t)$ and $S\psi_1(t)$ are disturbing moments about the axis of the outer gimbal and the axis of the housing, respectively, which arise, for example, as a consequence of the pitching or rolling of the ship on which the tracking system is mounted. Further, we will assume that $\psi_1(t)$ and $\psi_2(t)$ are stationary random processes with mathematical expectations equal to zero.

We denote by $-ly_2(t)$ and $Sy_1(t)$ the moments about the axis of the outer ring and the axis of the housing of the gyroscope, respectively, which are superimposed by the correcting electromagnets. These moments are proportional to the signals $y_2(t)$ and $y_1(t)$ which are formed in the converter according to the following law

$$\begin{aligned} y_1(t) &= X_{11}(D)[\theta_1(t) - \alpha(t)] + X_{12}(D)[\theta_2(t) - \beta(t)] \\ y_2(t) &= X_{21}(D)[\theta_1(t) - \alpha(t)] + X_{22}(D)[\theta_2(t) - \beta(t)] \end{aligned} \tag{1.2}$$

where

$$\theta_i(t) = m_i(t) + n_i(t) \qquad (i = 1, 2) \tag{1.3}$$

are the input signals into the system, $m_i(t)$ being the true signal and $n_i(t)$ the noise. The true signal $m_i(t)$ and the noise $n_i(t)$ are uncorrelated stationary random processes with mathematical expectations equal to zero.

We denote by $X_{jk}(D)$ $(j,\ k = 1,\ 2)$ elements of the transfer matrix function

$$X(D) = \| X_{jk}(D) \| \qquad \left(D = \frac{d}{dt}\right) \tag{1.4}$$

of the converter. As was mentioned above, the function $X(D)$ is to be determined from the condition of optimal reproduction of the true input signal by the system, in order that the angle of rotation of the external gimbal $\alpha(t)$ will be as close as possible to $m_1(t)$, and the angle of rotation of the housing of the gyroscope $\beta(t)$ will be as close as possible to $m_2(t)$.

Equations (1.1) may be put in the form

$$\alpha'' + \frac{\sigma}{A}\alpha' - \frac{H}{A}\beta' = -\frac{l}{A}[y_2(t) + \psi_2(t)]$$
$$\beta'' + \frac{H}{B}\alpha' = \frac{S}{B}[y_1(t) + \psi_1(t)] \tag{1.5}$$

We introduce the matrices

$$L(D) = \left\| \begin{matrix} D^2 + (\sigma/A)\,D & -(H/A)\,D) \\ (H/B)\,D & D^2 \end{matrix} \right\|, \qquad z(t) = \left\| \begin{matrix} \alpha(t) \\ \beta(t) \end{matrix} \right\| \tag{1.6}$$

$$e(D) = \left\| \begin{matrix} 0 & -l/A \\ S/B & 0 \end{matrix} \right\|, \qquad y(t) = \left\| \begin{matrix} y_1(t) \\ y_2(t) \end{matrix} \right\|, \qquad \psi(t) = \left\| \begin{matrix} \psi_1(t) \\ \psi_2(t) \end{matrix} \right\| \tag{1.7}$$

Then the system (1.5) may be replaced by the matrix equation

$$L(D)\,z(t) = e(D)\,[y(t) + \psi(t)] \tag{1.8}$$

From Equation (1.8) it follows that

$$z(t) = Y(D)\,y(t) + Y(D)\,\psi(t) \qquad \left(Y(D) = \frac{L^*(D)\,e(D)}{\Delta(D)}\right) \tag{1.9}$$

where $L^*(D)$ is the adjoint of the matrix $L(D)$ and $\Delta(D)$ is the determinant of the matrix $L(D)$.

In order to represent in matrix form the relations (1.2), which describe the formation of signals in the converter, we introduce the matrices

$$\theta(t) = \left\| \begin{matrix} \theta_1(t) \\ \theta_2(t) \end{matrix} \right\|, \qquad m(t) = \left\| \begin{matrix} m_1(t) \\ m_2(t) \end{matrix} \right\|, \qquad n(t) = \left\| \begin{matrix} n_1(t) \\ n_2(t) \end{matrix} \right\| \tag{1.10}$$

In accordance with (1.2), (1.4), (1.6) and (1.7) we obtain the following relation

$$y(t) = X(D)\,[\theta(t) - z(t)] \tag{1.11}$$

Since according to (1.3) and (1.10)

$$\theta(t) = m(t) + n(t) \tag{1.12}$$

where $m(t)$ is the matrix of the true signal and $n(t)$ the disturbance matrix, the relation (1.11) takes the form

$$y(t) = X(D)\,[m(t) + n(t)] - X(D)\,z(t) \tag{1.13}$$

Substituting into (1.9) the value of $y(t)$ according to (1.13), we obtain the matrix differential equation describing the tracking system together with the converter

$$[E+Y(D)X(D)]z(t)=Y(D)X(D)m(t)+Y(D)X(D)n(t)+Y(D)\psi(t) \quad (1.14)$$

Here E denotes the unit matrix. We now denote by $Z(D)$ the inverse of the matrix $E + Y(D)X(D)$:

$$Z(D)=[E+Y(D)X(D)]^{-1} \quad (1.15)$$

We premultiply Equation (1.14) by the matrix $Z(D)$

$$z(t)=Z(D)Y(D)X(D)m(t)+Z(D)Y(D)X(D)n(t)+Z(D)Y(D)\psi(t) \quad (1.16)$$

We denote by $\varepsilon(t)$ the error matrix in the reproduction of the true signals $m_1(t)$ and $m_2(t)$.

$$\varepsilon(t)=m(t)-z(t) \quad (1.17)$$

Since according to (1.15)

$$Z(D)[E+Y(D)X(D)]=E \quad (1.18)$$

we have the identity

$$m(t)=Z(D)m(t)+Z(D)Y(D)X(D)m(t) \quad (1.19)$$

Substituting $m(t)$ and $z(t)$ from (1.19) and (1.16) into (1.17), we reduce the error matrix to the form

$$\varepsilon(t)=Z(D)[m(t)-Y(D)\psi(t)-Y(D)X(D)n(t)] \quad (1.20)$$

In Expression (1.20) the matrix transfer function of the converter $X(D)$, which is to be determined, appears explicitly, as well as through $Z(D)$.

2. Introduction of disturbing forces into the input of the converter. For solution of the problem when disturbing forces $\psi_1(t)$ and $\psi_2(t)$ are introduced into the input of the converter, it is necessary to put the tracking system described in Section 1 into correspondence with another tracking system having no disturbing forces, but with an input signal

$$\theta^*(t)=M(t)+N(t) \quad (2.1)$$

where $M(t) = \| M_i(t) \|$ is the matrix of the true signal, and $N(t) = \| N_i(t) \|$ is the disturbance matrix, chosen so that the matrix error

$\varepsilon(t)$ of reproduction of the signal $M(t)$ coincides identically with the matrix $\varepsilon(t)$ determined by the Expression (1.20).

For the system considered in this section, the Expression (1.9) defining the output signal of the system must be replaced by the following expression

$$z^*(t) = Y(D)\, y^*(t) \tag{2.2}$$

where the starred symbols refer to the new tracking system.

It follows from (1.13) and (2.1) that the output signal of the converter now takes the form

$$y^*(t) = X(D)\,[M(t) + N(t)] - X(D)\, z^*(t) \tag{2.3}$$

From (2.2) and (2.3) it follows that

$$[E + Y(D)\, X(D)]\, z^*(t) = Y(D)\, X(D)\, M(t) + Y(D)\, X(D)\, N(t) \tag{2.4}$$

where, as above, E denotes the unit matrix.

Premultiplying the left- and right-hand sides of Equation (2.4) by the matrix $Z(D)$ defined in (1.15), we have

$$z^*(t) = Z(D)\, Y(D)\, X(D)\, M(t) + Z(D)\, Y(D)\, X(D)\, N(t) \tag{2.5}$$

The error $\varepsilon^*(t)$ in reproducing the signal $M(t)$ is

$$\varepsilon^*(t) = M(t) - z^*(t) \tag{2.6}$$

By analogy with (1.19)

$$M(t) = Z(D)\, M(t) + Z(D)\, Y(D)\, X(D)\, M(t) \tag{2.7}$$

Substituting Expressions (2.7) and (2.5) into (2.6), we obtain

$$\varepsilon^*(t) = Z(D)\, M(t) - Z(D)\, Y(D)\, X(D)\, N(t) \tag{2.8}$$

From comparison of the Expressions (2.8) and (1.20) it is clear that in order to have the identity

$$\varepsilon^*(t) \equiv \varepsilon(t) \tag{2.9}$$

we must take

$$M(t) = m(t) - Y(D)\, \psi(t), \qquad N(t) = n(t) \tag{2.10}$$

The signals obtained from (2.10) may be called the reduced input signals.

In accordance with (2.10) and (2.1), the matrix of the input signals $\theta^*(t)$ takes the form

$$\theta^*(t) = \begin{Vmatrix} \theta_1^*(t) \\ \theta_2^*(t) \end{Vmatrix} = \begin{Vmatrix} m_1(t) - Y_{11}(D)\psi_1(t) - Y_{12}(D)\psi_2(t) + n_1(t) \\ m_2(t) - Y_{21}(D)\psi_1(t) - Y_{22}(D)\psi_2(t) + n_2(t) \end{Vmatrix} \qquad (2.11)$$

As is clear from (2.11), the reduced input signal matrix $\theta^*(t)$ is a two-dimensional random process, the elements of which $\theta_1^*(t)$ and $\theta_2^*(t)$ are correlated even in the case when

$$\theta_1(t) = m_1(t) + n_1(t), \qquad \theta_2(t) = m_2(t) + n_2(t)$$

are uncorrelated.

In order to treat the problem of minimizing the root-mean-square error $\sqrt{\varepsilon^2}$ of reproduction, it is more convenient to proceed from the scheme outlined in Section 2, since in this it is possible to apply directly the method of Wiener, as will be shown below.

3. Equivalent system without feedback. We now consider a system without feedback, i.e. a filter into the input of which is fed the signal (2.1)

$$\theta^*(t) = M(t) + N(t)$$

where $M(t)$ and $N(t)$ are defined by Expressions (2.10). We assume that $M(t)$ is the true signal, which the filter must reproduce, while $N(t)$ is noise. The transfer function matrix of the filter is denoted by $\Phi(D)$. The output signal from the filter is the following

$$z^{**}(t) = \Phi(D)[M(t) + N(t)] \qquad (3.1)$$

The error in reproducing the signal $M(t)$ is

$$\varepsilon^{**}(t) = M(t) - z^{**}(t) \qquad (3.2)$$

or, in accordance with (3.1)

$$\varepsilon^{**}(t) = [E - \Phi(D)]M(t) - \Phi(D)N(t) \qquad (3.3)$$

where as above E is the unit matrix.

Let $\Phi(D)$ be the optimal transfer function of the filter, giving the minimum root-mean-square error

$$\sqrt{\overline{\varepsilon_j^{**2}}} \qquad (j = 1, 2)$$

in reproducing the signals $M_j(t)$ $(j = 1, 2)$. The determination of the function $\Phi(D)$ will be carried out below.

According to (2.9) $\varepsilon(t) \equiv \varepsilon^*(t)$; hence the optimal reproduction of the true signal $m_j(t)$ $(j = 1, 2)$ in the original problem (Section 1) will occur when the condition

$$\varepsilon^{**}(t) \equiv \varepsilon^*(t) \tag{3.4}$$

is satisfied, where $\varepsilon^*(t)$ is defined by Expression (2.8).

It is not difficult to show that condition (3.4) will be satisfied if the matrix $X(D)$ is chosen so that it satisfies the relation

$$Z(D)\,Y(D)\,X(D) = \Phi(D) \tag{3.5}$$

In fact, according to (1.18)

$$Z(D) = E - Z(D)\,Y(D)\,X(D) \tag{3.6}$$

In accordance with (3.6) and (3.5), the expression (2.8) takes the form

$$\varepsilon^*(t) = [E - \Phi(D)]\,M(t) - \Phi(D)\,N(t) \tag{3.7}$$

i.e. it agrees completely with the Expression (3.3) for $\varepsilon^{**}(t)$.

The unknown matrix $X(D)$ remains to be determined from relation (3.5). For this we premultiply the left- and right-hand sides of relation (3.5) by the matrix $E + Y(D)X(D)$. Taking into account (1.18), we have

$$Y(D)\,X(D) = [E + Y(D)\,X(D)]\,\Phi(D) \tag{3.8}$$

whence

$$Y(D)\,X(D)\,[E - \Phi(D)] = \Phi(D) \tag{3.9}$$

and consequently the desired transfer function matrix $X(D)$ of the converter has the form

$$X(D) = Y^{-1}(D)\,\Phi(D)\,[E - \Phi(D)]^{-1} \tag{3.10}$$

4. An alternate method of conversion. We consider the possibility of using a computer in order to realize the converter with the transfer function matrix $X(D)$ given by the Expression (3.10). We introduce the notation

$$r_1(t) = \theta_1(t) - \alpha(t), \qquad r_2(t) = \theta_2(t) - \beta(t) \tag{4.1}$$

Equations (1.2), which govern the operation of the converter, are reduced to the form

$$y_1(t) = X_{11}(D)\, r_1(t) + X_{12}(D)\, r_2(t),\ \ y_2(t) = X_{21}(D)\, r_1(t) + X_{22}(D)\, r_2(t) \quad (4.2)$$

Introducing the matrix

$$r(t) = \begin{Vmatrix} r_1(t) \\ r_2(t) \end{Vmatrix} \quad (4.3)$$

and noting that according to (1.10) and (1.7)

$$r(t) = \theta(t) - z(t) \quad (4.4)$$

we replace Equations (1.2) with the matrix differential equation

$$y(t) = X(D)\, r(t) \quad (4.5)$$

Substituting into (4.5) the Expression (3.10) found for $X(D)$, we obtain

$$y(t) = Y^{-1}(D)\, \dot{\Phi}(D)\, [E - \Phi(D)]^{-1} r(t) \quad (4.6)$$

From (4.6) it follows that

$$Y(D)\, y(t) = \Phi(D)\, [E - \Phi(D)]^{-1}\, r(t) \quad (4.7)$$

Denoting by $\zeta(t)$ the function

$$\zeta(t) = [E - \Phi(D)]^{-1} r(t) \quad (4.8)$$

we find that

$$\zeta(t) - \Phi(D)\, \zeta(t) = r(t) \quad (4.9)$$

We denote by $\Gamma(t)$ the weighting function of the optimal filter, i.e. the filter with the transfer function $\Phi(D)$. As is known, these functions are linked through the operational relation

$$p\Phi(p) \div\!\!\rightarrow \Gamma(t) \quad (4.10)$$

i.e. the function $p\,\Phi(p)$ is the Carson-Heaviside transform of the original function $\Gamma(t)$. With the help of (4.10) we may pass from (4.9) to the integral equation

$$\zeta(t) - \int_0^t \Gamma(t-\tau)\, \zeta(\tau)\, d\tau = r(t) \quad (4.11)$$

for the unknown function $\zeta(t)$.

We will assume that a computer for the solution of Equation (4.11) is a part of the converter. If the solution of (4.11) is fed into the input of the filter having the transfer function $\Phi(D)$, then from the filter output we get the signal

$$\xi(t) = \Phi(D)\zeta(t) \tag{4.12}$$

Comparing Expressions (4.8), (4.12) and (4.7), we find that

$$Y(D)y(t) = \xi(t) \tag{4.13}$$

In the Expression (4.13) $\xi(t)$ is a known function, whereas the function $y(t)$ is to be determined. The function $Y(D)$ is, according to (1.9), the transfer function of the gyroscope. Denoting the weighting function of the gyroscope by $W(t)$, i.e. letting

$$pY(p) \doteq W(t) \tag{4.14}$$

we pass from relation (4.13) to the integral equation for the unknown function $y(t)$

$$\int_0^t W(t-\tau)\, y(\tau)\, dt = \xi(t) \tag{4.15}$$

The solution of the integral equation (4.15) also represents the signal $y(t)$ which must come from the converter into the input of the gyroscope.

Thus the tracking system converter, which has a transfer function $X(D)$, consists of a sequence of three coupled devices: a computer for solving the integral equation (4.11), a filter with the transfer function $\Phi(D)$ into which is fed the solution of the integral equation (4.11), and a computer for solving the integral equation (4.15). This alternate form of the converter has an advantage in that when the form of the input signals $\theta_1^*(t)$ and $\theta_2^*(t)$ is changed it is necessary only to replace the optimal filter and to modify the kernel $\Gamma(t-\tau)$ in the Equation (4.11).

5. Determination of the transfer function $\Phi(D)$ of the optimal filter. The weighting function matrix

$$\Gamma(t) = \begin{Vmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{Vmatrix} \tag{5.1}$$

of the optimal filter considered in Section 3 must satisfy the integral equation obtained by Wiener [3]

$$\int_0^\infty R(\tau_2-\tau_1)\,\Gamma'(\tau_1)\,d\tau_1 = U(\tau_2) \quad \text{for } \tau_2 \geqslant 0 \tag{5.2}$$

and the condition

$$\Gamma(t) = 0 \quad \text{for } t < 0 \tag{5.3}$$

Here $\Gamma'(t)$ is the transpose of the matrix $\Gamma(t)$, and $R(\tau)$ and $U(\tau)$ denote the matrices

$$R(\tau) = \begin{Vmatrix} R_{11}(\tau) & R_{12}(\tau) \\ R_{21}(\tau) & R_{22}(\tau) \end{Vmatrix}, \qquad U(\tau) = \begin{Vmatrix} U_{11}(\tau) & U_{12}(\tau) \\ U_{21}(\tau) & U_{22}(\tau) \end{Vmatrix} \tag{5.4}$$

where $R_{ij}(\tau)$ $(i, j = 1, 2)$ are the correlation functions of the random processes $\theta_i^*(t)$ and $\theta_j^*(t)$, and $U_{ij}(\tau)$ $(i, j = 1, 2)$ are the correlation functions of the random processes $M_i(t)$ and $\theta_j^*(t)$.

We denote the spectral density matrices of $R(\tau)$ and $U(\tau)$ by $G^{(R)}(\omega)$ and $G^{(U)}(\omega)$, respectively.

$$G^{(R)}(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau} d\tau, \qquad G^{(U)}(\omega) = \int_{-\infty}^{\infty} U(\tau)\, e^{-i\omega\tau} d\tau \tag{5.5}$$

The variance of error in the reproduction of the signal $M_j(t)$ $(j = 1, 2)$ by the optimal filter will be determined by the expression [3]

$$\overline{\varepsilon_j^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{M_jM_j}(\omega)\, d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{k=1}^{2} \sum_{\mu=1}^{2} \Phi_{jk}(-i\omega)\, \Phi_{j\mu}(i\omega)\, G_{k\mu}{}^{(R)}(\omega)\, d\omega \qquad (j = 1, 2) \tag{5.6}$$

where

$$\Phi_{jk}(i\omega) = \int_{-\infty}^{\infty} \Gamma_{jk}(\tau)\, e^{-i\omega\tau}\, d\tau \tag{5.7}$$

and $G_{M_jM_j}(\omega)$ $(j = 1, 2)$ denotes the spectral density of the random process $M_j(t)$.

The matrix integral equation (5.2) and the condition (5.3) are equivalent to a system of scalar integral equations

$$\sum_{k=1}^{2} \int_{0}^{\infty} \Gamma_{jk}(\tau_1)\, R_{\mu k}(\tau_2 - \tau_1)\, d\tau_1 - U_{\mu j}(\tau_2) = 0 \quad \text{for } \tau_2 \geqslant 0 \qquad (j, \mu = 1, 2) \tag{5.8}$$

and the conditions

$$\Gamma_{jk}(t) = 0 \quad \text{for } t < 0 \tag{5.9}$$

In order to obtain in place of (5.8) equations valid for arbitrary values of τ_2, both positive and negative, we introduce functions $f_{j\mu}(\tau_2)$ (j, $\mu = 1, 2$), defined by the relations

$$f_{j\mu}(\tau_2) = \sum_{k=1}^{2} \int_0^\infty \Gamma_{jk}(\tau_1) R_{\mu k}(\tau_2 - \tau_1)\, d\tau_1 - U_{\mu j}(\tau_2) \quad \text{for } \tau_2 < 0 \tag{5.10}$$

$$f_{j\mu}(\tau_2) = 0 \quad \text{for } \tau_2 \geqslant 0$$

With the aid of (5.8) and (5.10) we obtain a system of integral equations, valid for arbitrary values of τ_2

$$\sum_{k=1}^{2} \int_0^\infty \Gamma_{jk}(\tau_1) R_{\mu k}(\tau_2 - \tau_1)\, d\tau_1 - U_{\mu j}(\tau_2) = f_{j\mu}(\tau_2) \qquad (j, \mu = 1, 2) \tag{5.11}$$

Multiplying the left- and right-hand sides of equations (5.11) by $e^{-i\omega\tau_2}$ and integrating them with respect to τ_2 between the limits $-\infty$ and ∞, we obtain

$$\sum_{k=1}^{2} \Phi_{jk}(i\omega) G_{\mu k}^{(R)}(\omega) - G_{\mu j}^{(U)}(\omega) = F_{j\mu}^{-}(\omega) \qquad (j, \mu = 1, 2) \tag{5.12}$$

where

$$F_{j\mu}^{-}(\omega) = \int_{-\infty}^{\infty} f_{j\mu}(\tau) e^{-i\omega\tau}\, d\tau \tag{5.13}$$

We note that $f_j(\tau_2) = 0$ for $\tau_2 \geqslant 0$, in accordance with (5.10); hence the function $F_{j\mu}^{-}(\omega)$ has no poles in the upper half of the ω-plane.

The system of scalar equations (5.12) may be represented in matrix form as

$$\Phi(i\omega) G^{(R)'}(\omega) - G^{(U)'}(\omega) = F^{-}(\omega) \tag{5.14}$$

where $G^{(R)'}(\omega)$ and $G^{(U)'}(\omega)$ are the transposes of the matrices $G^{(R)}(\omega)$ and $G^{(U)}(\omega)$, respectively. From (5.14) it follows that

$$\Phi(i\omega) = G^{(U)'}(\omega) [G^{(R)'}(\omega)]^{-1} + F^{-}(\omega) [G^{(R)'}(\omega)]^{-1} \tag{5.15}$$

where the symbol $[\ ]^{-1}$ denotes the inverse matrix.

We denote by Ω_{jk} the cofactor of $G_{jk}^{(R)}(\omega)$ in the determinant of the $G^{(R)}(\omega)$. The matrix $\Omega = \| \Omega_{jk} \|$ will have the following form

$$\Omega = \begin{Vmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{Vmatrix} = \begin{Vmatrix} G_{22}^{(R)}(\omega) & -G_{21}^{(R)}(\omega) \\ -G_{12}^{(R)}(\omega) & G_{11}^{(R)}(\omega) \end{Vmatrix} \tag{5.16}$$

By G we denote the determinant of the matrix $G^{(R)}(\omega)$

$$G = \det G^{(R)}(\omega) \tag{5.17}$$

With the help of (5.16) and (5.17) we obtain

$$[G^{(R)'}(\omega)]^{-1} = \frac{\Omega}{G} \tag{5.18}$$

The Expression (5.15) now takes the form

$$\Phi(i\omega) = \frac{1}{G}\,[G^{(U)'}(\omega) + F^{-}(\omega)]\,\Omega \tag{5.19}$$

The elements of the matrix $\Phi(i\omega)$ become

$$\Phi_{jk}(i\omega) = \frac{1}{G}\sum_{\mu=1}^{2}[G_{\mu j}^{(U)}(\omega) + F_{j\mu}^{-}(\omega)]\,\Omega_{\mu k} \qquad (j,\ k = 1,\ 2) \tag{5.20}$$

In the case where the spectral densities $G_{ij}^{(R)}(\omega)$ are rational functions of ω, the determinant G of the matrix $G^{(R)}(\omega)$ may be represented in the form

$$G = G^{+}G^{-} \tag{5.21}$$

where G^{+} and G^{-} are complex conjugate functions, all zeros and poles of the function G^{+} being situated in the upper half-plane, and all zeros and poles of the function G^{-} in the lower half of the complex ω-plane.

With the help of (5.21) we reduce Expression (5.20) to the form

$$G^{+}\Phi_{jk}(i\omega) = \frac{1}{G^{-}}\sum_{\mu=1}^{2} G_{\mu j}^{(U)}(\omega)\,\Omega_{\mu k} + \frac{1}{G^{-}}\sum_{\mu=1}^{2} F_{j\mu}^{-}(\omega)\,\Omega_{\mu k} \qquad (j,\ k=1,\ 2) \tag{5.22}$$

Expanding the elementary fraction, we can represent the first term on the right-hand side of (5.22) in the form

$$\frac{1}{G^{-}}\sum_{\mu=1}^{2} G_{\mu j}^{(U)}(\omega)\,\Omega_{\mu k} = T_{jk}^{+}(\omega) + T_{jk}^{-}(\omega) \qquad (j,\ k = 1,\ 2) \tag{5.23}$$

where all poles of the function $T_{jk}^{+}(\omega)$ are located in the upper half-plane, and those of the function $T_{jk}^{-}(\omega)$ are located in the lower half of the complex ω-plane. The expansion (5.23) may be carried out easily, since the functions under consideration are completely known.

We write the second term on the right-hand side of (5.22) as follows

$$\frac{1}{G^{-}}\sum_{\mu=1}^{2} F_{j\mu}^{-}(\omega)\,\Omega_{\mu k} = \sum_{i}\sum_{l=1}^{q_i}\frac{C_{il}^{(jk)}}{(\omega-\gamma_i)^{l}} + P_{jk}^{-} \qquad (j,\ k = 1,\ 2) \tag{5.24}$$

where P_{jk}^{-} is a function having all its poles in the lower half of the complex ω-plane, and γ_i are the poles of the functions Ω_{1k} and Ω_{2k}, which are located in the upper half of the ω-plane; q_i denotes the multiplicity of these poles. Since we know only that the functions $F_{j\mu}^{-}(\omega)$ have no poles in the upper half of the ω-plane while the functions themselves are unknown, then for the time being the coefficients $C_{il}^{(jk)}$ remain undetermined.

Since $\Gamma_{jk}(t) = 0$ (j, $k = 1, 2$) for $t < 0$, then the functions $\Phi_{jk}(i\omega)$ have no poles in the lower half of the ω-plane, and consequently the left-hand side of (5.22) is a function having all of its poles in the upper half of the ω-plane.

Therefore the desired transfer functions $\Phi_{jk}(i\omega)$ (j, $k = 1, 2$) are, from (5.22), (5.23) and (5.24)

$$\Phi_{jk}(i\omega) = \frac{1}{G^{+}}\left[T_{jk}^{+}(\omega) + \sum_i \sum_{l=1}^{q_i} \frac{C_{il}^{(jk)}}{(\omega-\gamma_i)^l}\right] \qquad (j,\ k = 1,\ 2) \qquad (5.25)$$

The determination of the unknown coefficients $C_{il}^{(jk)}$ is implemented by substituting the functions $\Phi_{jk}(i\omega)$ found above into Equation (5.12) and finding all the poles of the functions entering in (5.12) which are located in the upper half of the complex ω-plane. Thus we arrive at relations of the form

$$\left[\sum_{k=1}^{2} \Phi_{jk}(i\omega)\, G_{\mu k}^{(R)}(\omega)\right]^{+} = [G_{\mu j}^{(U)}(\omega)]^{+} \qquad (j,\ \mu = 1,\ 2) \qquad (5.26)$$

where the symbol $[\ ]^{+}$ denotes functions generated in a similar manner as the functions $T_{jk}^{+}(\omega)$ in Expression (5.23).

Equating the residues at the corresponding singular points of the functions on the left- and right-hand sides of (5.26), we obtain a system of linear algebraic equations for the coefficients $C_{ij}^{(jk)}$, which then are found in the usual way.

6. Example. As an example we consider the case where the spectral density of the incoming signal has the form

$$G_{m_i m_i} = \frac{2\varkappa_i \chi_i}{\omega^2 + \varkappa_i^2}, \qquad G_{n_i n_i} = K_i \qquad (i = 1,\ 2) \qquad (6.1)$$

The random processes $m_i(t)$ and $n_i(t)$ ($i = 1, 2$) are uncorrelated.

We assume that the disturbing moment about the axis of the inner gimbal of the gyroscope is equal to zero

$$\psi_1(t) \equiv 0 \tag{6.2}$$

In order to determine the disturbing moment about the axis of the outer gimbal of the gyroscope we note that the first of Equations (1.1), including the frictional forces in the bearings of the outer gimbal, takes the form

$$A\alpha'' - H\beta' = -ly_2(t) - \sigma(\alpha' - \vartheta') \tag{6.3}$$

where ϑ' is the angular velocity of vibration of the object (pitch or roll of the vessel) on which the tracking system is mounted. Comparing Equation (6.3) with the first of Equations (1.1) we see that

$$\psi_2(t) = -\frac{\sigma}{l} D\vartheta \qquad \left(D = \frac{d}{dt}\right) \tag{6.4}$$

where ϑ is the angle of roll of the object, which we assume to be a stationary random process with spectral density

$$G_{\vartheta\vartheta} = Q \tag{6.5}$$

According to (2.10), (2.11) and (6.2) the input signal $M_i(t)$ $(i=1,2)$ becomes

$$M_i(t) = m_i(t) + m_i^*(t) \qquad (i = 1,\ 2) \tag{6.6}$$

where

$$m_1^*(t) = -Y_{12}(D)\psi_2(t), \qquad m_2^*(t) = -Y_{22}(D)\psi_2(t) \tag{6.7}$$

Here $Y_{jk}(D)$ are the elements of the gyroscope transfer function matrix $Y(D)$, which is defined by the Expression (1.9). In accordance with (6.7) and (1.9)

$$\begin{aligned} m_1^*(t) &= \frac{-\sigma_1 D}{D^2 + \sigma_1 D + q^2}\vartheta(t) \\ m_2^*(t) &= \frac{\sigma_1 H/B}{D^2 + \sigma_1 D + q^2}\vartheta(t) \end{aligned} \qquad \left(\sigma_1 = \frac{\sigma}{A},\ q^2 = \frac{H^2}{AB}\right) \tag{6.8}$$

In order to obtain the matrix of the spectral densities of the random processes $m_1^*(t)$ and $m_2^*(t)$ we note that since according to (6.8)

$$m_1^*(t) = -(B/H)Dm_2^*(t) \tag{6.9}$$

then the mutual spectral densities become

$$G_{m_2^*m_1^*} = -G_{m_1^*m_2^*} = -i\omega(B/H)G_{m_2^*m_2^*} \tag{6.10}$$

Thus the matrix of the spectral densities of the random processes $m_1^*(t)$ and $m_2^*(t)$ will have the form

$$G^{(m^*)}(\omega) = \begin{Vmatrix} \dfrac{\sigma_1^2\omega^2 Q}{(\omega^2 - q^2)^2 + \sigma_1^2\omega^2} & \dfrac{i\omega\sigma_1^2 QH/B}{(\omega^2 - q^2)^2 + \sigma_1^2\omega^2} \\ \dfrac{-i\omega\sigma_1^2 QH/B}{(\omega^2 - q^2)^2 + \sigma_1^2\omega^2} & \dfrac{\sigma_1^2 QH^2/B^2}{(\omega^2 - q^2)^2 + \sigma_1^2\omega^2} \end{Vmatrix} \tag{6.11}$$

The spectral densities of the random processes $M_1(t)$ and $M_2(t)$ in (5.6) become

$$G_{M_1M_1} = G_{m_1m_1} + G_{m_1^*m_1^*}, \qquad G_{M_2M_2} = G_{m_2m_2} + G_{m_2^*m_2^*} \tag{6.12}$$

The matrices of the spectral densities $G^{(R)}(\omega)$ and $G^{(U)}(\omega)$, which are defined by Expressions (5.5), become, according to (2.10) and (6.6)

$$G^{(R)}(\omega) = \begin{Vmatrix} G_{m_1m_1} + G_{m_1^*m_1^*} + G_{n_1n_1} & G_{m_1^*m_2^*} \\ G_{m_2^*m_1^*} & G_{m_2m_2} + G_{m_2^*m_2^*} + G_{n_2n_2} \end{Vmatrix} \tag{6.13}$$

$$G^{(U)}(\omega) = \begin{Vmatrix} G_{m_1m_1} + G_{m_1^*m_1^*} & G_{m_1^*m_2^*} \\ G_{m_2^*m_1^*} & G_{m_2m_2} + G_{m_2^*m_2^*} \end{Vmatrix} \tag{6.14}$$

We take the following values for the parameters of the spectral densities given in Expressions (6.1) and (6.2)

$$\chi_1 = 0.01, \quad \kappa_1 = 0.5 \text{ sec}^{-1}, \quad \chi_2 = 0.02, \quad \kappa_2 = 0.3 \text{ sec}^{-1}$$

$$K_1 = 16.10^{-6}, \quad K_2 = 9.10^{-6}, \quad Q = 10^{-8}$$

For the parameters of the gyroscope we take

$$H/A = 2.5 \text{ sec}^{-1}, \quad H/B = 1000 \text{ sec}^{-1}, \quad l/A = 2,5 \text{ sec}^{-2}$$

$$S/B = 100 \text{ sec}^{-2}, \quad \sigma_1 = \sigma/A = 10 \text{ sec}^{-1}$$

For these values the frequency of nutational oscillations of the gyroscope is $q = 50 \text{ sec}^{-1}$.

For $Q = 0$, by (5.25) the optimal transfer function $\Phi(D)$ becomes

$$\Phi(D) = \begin{Vmatrix} 24.5\,(D + 25)^{-1} & 0 \\ 0 & 36\,(D + 36.6)^{-1} \end{Vmatrix} \tag{6.15}$$

For the matrix $\Phi(D)$ given here we can find with the help of (3.10) the transfer function matrix $X(D)$ of the converter. Denoting

$$\Xi(D) = \Phi(D)\,[E - \Phi(D)]^{-1} \tag{6.16}$$

we have

$$\Xi(D) = \begin{Vmatrix} 24.5\,(D + 0,5)^{-1} & 0 \\ 0 & 36\,(D + 0.6)^{-1} \end{Vmatrix} \tag{6.17}$$

The matrix $X(D)$ by virtue of (3.10) becomes

$$X(D) = \left\| \begin{matrix} \Xi_{11}(H/S)D & \Xi_{22}(B/S)D^2 \\ -\Xi_{11}(A/l)(D+\sigma/A)D & \Xi_{22}(H/l)D \end{matrix} \right\| \qquad (6.18)$$

where Ξ_{11} and Ξ_{22} are the elements of the matrix (6.17).

For the gyroscope parameters quoted above the matrix $X(D)$ takes the form

$$X(D) = \left\| \begin{matrix} 245D(D+0.5)^{-1} & 0.36D^2(D+0.6)^{-1} \\ -9.8D(D+10)(D+0.5)^{-1} & 36D(D+0.6)^{-1} \end{matrix} \right\| \qquad (6.19)$$

From (5.6) the variance of the error in reproducing the true signal is

$$\overline{\varepsilon_1^2} = 4\cdot 10^{-4}, \qquad \overline{\varepsilon_2^2} = 4.9\cdot 10^{-4}$$

The root-mean-square value of the error of reproduction is

$$\sqrt{\overline{\varepsilon_1^2}} = 2\cdot 10^{-2}, \qquad \sqrt{\overline{\varepsilon_2^2}} = 2.21\cdot 10^{-2}$$

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